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Peristaltic flow of viscoelastic liquids

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The mechanism of peristaltic transport of an incompressible viscoelastic fluid by means of an infinite train of sinusoidal waves travelling along the wall of the duct is studied in the case of a plane flow. The main assumptions are that the relevant Reynolds number is small enough to neglect inertia forces, and that the ratio of the wavelength and the channel height is large, which implies that the pressure is constant over the cross-section. For sufficiently small values of the ratio of the wave amplitude and the mean height of the channel, details of the fluid motion are studied analytically within a second-order approximation with respect to the amplitude ratio. Under these conditions the integral constitutive equation of finite linear viscoelasticity is relevant. Particular attention is given to the pressure–discharge characteristics of the peristaltic pump and to the pumping efficiency. The results are influenced by specific values of the complex viscosity of the fluid, which can be determined using standard rheometers. In general, the rate of discharge turns out to be a non-monotonic function of the wave speed. This leads to an optimal wave speed, for which the memory of the fluid particles extends over several wave periods. From an energetic point of view, relatively small wave speeds are the best, where the fluid changes its state slowly such that the memory and with it the elasticity of the fluid do not influence the flow field at all. As the dimensionless memory parameter tends to zero, the analytical results reduce to the well-known case of a Newtonian fluid.

1. Introduction

Our purpose is to investigate the mechanism by which a fluid is transported through a duct when contraction waves propagate progressively along its wall. This valveless-pumping principle, which is called peristalsis, plays a role in many physiological processes with fluid transport and is also exploited in technology, e.g. in so-called 'roller pumps'. In the last 15 years many investigations on peristaltic flow of Newtonian fluids have been carried out. Rath (1980) has given a survey of this subject, with a probably complete summary of the bibliography until 1978.

Studying peristaltic flows, especially with a view to applications in biomechanics and physiology, one should consider real material properties of the fluid being transported and determine the essential departures from the results of the theories for Newtonian fluids. These investigations are, also, interesting for technological applications, e.g. in the field of polymer processing. In this regard there are only few contributions in the literature. The earliest ones date back to Raju & Devanathan (1972, 1974). They considered the motion of an inelastic power-law fluid and of a special viscoelastic fluid of differential type of grade two through a tube with sinusoidal corrugation of small amplitude propagating in the axial direction. Within their approach (linear theory with respect to the wave amplitude) the rate of discharge is independent of the elasticity of the fluid. Therefore the papers give no information about the question how the pressure–discharge characteristic of a peristaltic pump varies if a viscoelastic fluid is transported instead of a Newtonian one.

Recently Becker (1980) has studied a simple model for peristaltic pumping of a non-Newtonian fluid. Because of the special geometry of the pump only the viscometric viscosity function enters the theory; the contraction may be of any height desired. The numerical results refer to the model of a Prandtl–Eyring fluid.

In this connection also the paper of Shukla *et al.* (1980) is worth mentioning. It is related to those situations in the organism for which the viscosity of the fluid varies from the wall to the centre of the duct. In the special case of a constant viscosity, the analytical results reduce to those of Shapiro, Jaffrin & Weinberg (1969).

In the following we investigate a plane peristaltic flow, which is idealized in several respects. The fluid is assumed to be within a channel of average height a, where one wall is at rest. An infinitely long sinusoidal wavetrain with amplitude ϵa ($|\epsilon| < 1$), wavelength l and wave velocity c travels over the wall (cf. figure 1a). We assume that the particles in the wavy wall move strictly up and down. Our aim is to find out which rate of flow $\bar{Q}$ [m^2/s] per channel width will be produced by the motion of the wall in the time average. Besides a, ϵ, l and c the additional factors affecting $\bar{Q}$ are the average pressure rise Δp_l registered in advancing one wavelength in the direction of the flow (in negative x-direction), the density ρ and the viscous and elastic properties of the fluid, especially the zero-shear-rate viscosity η_0 and a characteristic memory time λ.

From the nine quantities $a, \epsilon, l, c, \bar{Q}, \Delta p_l, \rho, \eta_0, \lambda$ six independent dimensionless quantities can be formed. We select the following:

$$\text{flow-rate parameter} \quad \Phi \equiv \frac{\bar{Q}}{ac}; \tag{1}$$

$$\text{pressure parameter} \quad K \equiv \frac{a^2 \Delta p_l}{\eta_0 c l}; \tag{2}$$

$$\text{Reynolds number} \quad R \equiv \frac{\rho a c}{\eta_0}; \tag{3}$$

together with the length ratio a/l, which is sometimes called the 'wave order', the dimensionless relative wave amplitude ϵ and the memory parameter $\lambda\omega$, where

$$\omega \equiv 2\pi \frac{c}{l} \tag{4}$$

is the characteristic angular frequency for the flow field. The relation then reads

$$\Phi = \Phi(K, R, \frac{a}{l}, \epsilon^2, \omega\lambda). \tag{5}$$

Here we can already recognize that the rate of flow will be an even function with regard to ϵ, using the following symmetry argument. In the relative frame fixed with the wave, the channel height is described by the time-independent function

$$h(x) = a\left(1 + \epsilon \cos \frac{2\pi x}{l}\right). \tag{6}$$

A change of sign of ϵ would cause a displacement of the wavetrain by $\frac{1}{2}l$. Thereby the rate of flow would naturally be unaffected. So we infer that with low amplitude

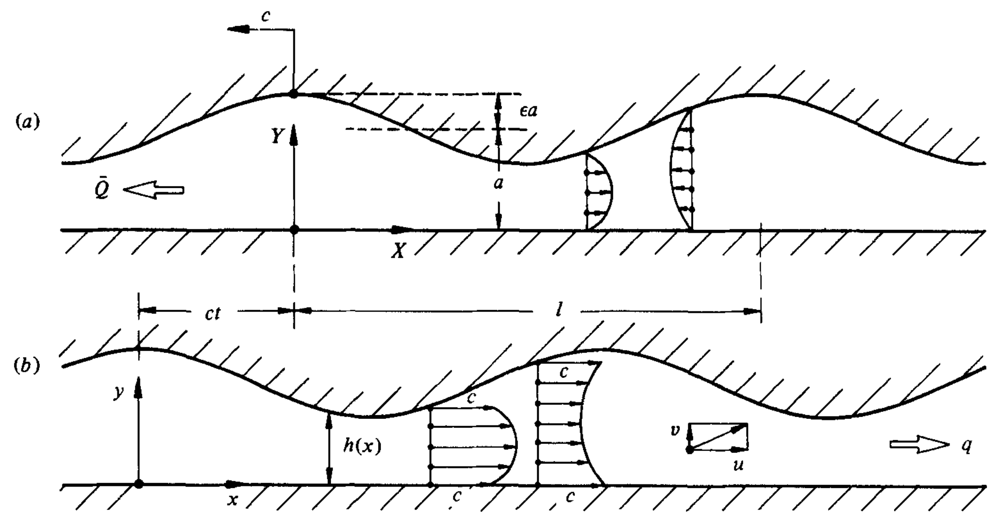

FIGURE 1. Plane peristaltic flow: (a) laboratory frame; (b) wave frame.

of the corrugation (for $|\epsilon| \ll 1$) at least an approximation of order ϵ^2 is necessary to cover the influence of the peristaltic motion on the flow rate.

In order to obtain analytical results it is essential to make some simplifying assumptions. In particular, we assume the wavelength to be sufficiently long compared with the channel height ($a/l \ll 1$). This will permit us to neglect quadratic terms of the 'slenderness' parameter a/l compared with those of order 1 in the analysis. Furthermore, we assume that the fluid is so viscous that inertia forces can be neglected compared with extra stresses ('creeping flow'). With regard to the dimensionless quantities this means that the Reynolds number reduced with the slenderness parameter, i.e. Ra/l (not R itself), has to be small compared with 1 (see Shapiro *et al.* 1969). As we neglect inertia forces completely, we consider the asymptotic case $Ra/l \to 0$. Both simplifications lead to the fact that the parameters R and a/l drop out in (5):

$$\Phi = \Phi(K, \epsilon^2, \omega\lambda). \tag{7}$$

According to Shapiro *et al.* (1969), in the case of a Newtonian fluid (with $\omega\lambda = 0$), the relation (7), which gives the pressure–discharge characteristic of the system, can be described by elementary functions in the limit of the approximations mentioned. The Newtonian results we need later on will be summarized in §3. If we consider arbitrary viscoelastic liquids, it will be necessary to assume furthermore that the relative wave amplitude ϵ is small, and we restrict ourselves to determining the right-hand side of (7) within a linear approximation in ϵ^2.

2. Equations of motion and boundary conditions

While the motion we investigate is unsteady viewed from the laboratory frame (with local coordinates X, Y) it is steady in the wave-fixed relative frame. Therefore we largely base our calculation on the wave frame (with local coordinates x, y and velocity components u, v; cf. figure 1).

Let the rate of flow per channel width in the wave frame be q (the positive direction is to the right). Then the flow rate Q in the laboratory frame (the positive direction is to the left) is related to q in the following way: $q = ch(X+ct) - Q(X+ct)$. For continuity reasons q is independent of x; whereas Q depends on $X+ct$. In the time average, which will be indicated by a bar, we get from this (note that $\bar{h} = a$)

$$\bar{Q} = ca - q. \tag{8}$$

The plane flow of an arbitrary incompressible fluid that is steady in the wave frame is described by the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

and two equations of motion which connect the components of acceleration with the pressure p and the elements $\tau_{xx}, \tau_{xy}, \tau_{yy}$ of the extra stress tensor. If we neglect terms of order $(a/l)^2$ compared with those of order 1, the equation of motion in the y-direction leads to the simple relation (see Böhme 1981)

$$-p + \tau_{yy} = -\tilde{p}(x). \tag{10}$$

Thus the slenderness condition implies that the normal stress in the direction perpendicular to the wall is constant over the cross-section of the channel and varies only along the direction parallel to the wall. This fulfils the balance of momentum in the y-direction, and it remains only to analyse the momentum balance in the x-direction. Neglecting inertia forces compared with friction forces, this presumes a sufficiently small reduced Reynolds number $Ra/l \to 0$. Under this assumption the relevant dynamic relation reads

$$\frac{\partial}{\partial x}(\tau_{xx} - \tau_{yy}) + \frac{\partial \tau_{xy}}{\partial y} = \frac{d\tilde{p}(x)}{dx}. \tag{11}$$

Concerning the boundary conditions of the flow field, from the no-slip condition at the two walls we find

$$u(x,0) = c, \quad v(x,0) = 0, \tag{12}$$

$$u(x,h(x)) = c, \quad v(x,h(x)) = c\frac{dh}{dx} = -\frac{2\pi a}{l} c\epsilon \sin\frac{2\pi x}{l}. \tag{13}$$

Moreover we take into account that the wall pressure rises in the negative x-direction over one wavelength by the given value Δp_l:

$$\Delta p_l \equiv -\int_0^l \frac{d\tilde{p}(x)}{dx} dx. \tag{14}$$

In the laboratory frame $\Delta p_l/l$ describes the mean pressure rise in that direction, where the wave moves and the fluid has to be transported.

3. The flow field of a Newtonian fluid

In the case of a Newtonian fluid with viscosity η_0, (11) reduces to the relation $\eta_0 \partial^2 u/\partial y^2 = d\tilde{p}/dx$. With the boundary conditions (12), (13) this gives the velocity field

$$u(x,y) = c + \frac{1}{2\eta_0}\frac{d\tilde{p}}{dx}[y^2 - h(x)\,y]. \tag{15}$$

By integration over the cross-section of the channel we obtain the flow rate q. Solving this relation for $d\tilde{p}/dx$ we get

$$\frac{d\tilde{p}}{dx} = 12\eta_0 c\frac{h-q/c}{h^3}. \tag{16}$$

If we use this expression in (14) and replace the constant q by $ca-\bar{Q}$ (equation (8)), we obtain a linear relation between pressure rise and rate of flow, which, in the case of the special function (6), takes the normalized form

$$K = \frac{6}{(1-\epsilon^2)^{\frac{5}{2}}}[3\epsilon^2-(2+\epsilon^2)\Phi]. \tag{17}$$

This formula corresponds to equation (13) of Shapiro *et al.* (1969). (Because their symbol a denotes half of the channel height, the factor $\frac{3}{4}$ appears there instead of 6 ahead of the bracket in our formula.) Equation (17) can be written compactly in the form $K/K_0 = 1-\Phi/\Phi_0$, where $\Phi_0 = 3\epsilon^2/(2+\epsilon^2)$ describes the normalized rate of flow without counterpressure, and $K_0 = 18\epsilon^2/(1-\epsilon^2)^{\frac{5}{2}}$ is the pressure parameter in the special situation where the flow rate vanishes. Henceforth we are only interested in real pumping situations, where the peristaltic flow is in the same direction as the pressure rise; this region is defined by the inequalities $0 \leqslant \Phi \leqslant \Phi_0, 0 \leqslant K \leqslant K_0$. We recognize that for small amplitude of the corrugation ($\epsilon^2 \ll 1$) the pressure parameter K has to be restricted to values of order ϵ^2, and the flow-rate parameter Φ then also is of order ϵ^2. This fact will be used later on.

First we consider the special case where the pressure difference over one wavelength disappears (no counterpressure, $K = 0$). In the Newtonian case under these conditions the pressure field $\tilde{p}(x)$ is influenced by the viscosity of the fluid, while the velocity field is not. A thorough consideration leads to the conclusion that, even in viscoelastic fluids, where the extra stresses are linearly connected with the velocity field, the velocity field of the creeping peristaltic flow is completely independent of the material properties in the special case $K = 0$. Thus it is identical with the previously considered Newtonian flow field. Because we intend to make use of this knowledge later on under the assumption of weak corrugation, we note here the Newtonian flow field in a first-order approximation with respect to ϵ (for which we will use the term 'ϵ^1 approximation'). By putting (6) and (16) into (15) and expanding on ascending powers of ϵ, one finds for the flow without global pressure rise ($K = 0$ and therefore $q/ac = 1+O(\epsilon^2)$ according to (8), (17))

$$\frac{u(x,y)}{c} = 1+6\epsilon\left(\frac{y^2}{a^2}-\frac{y}{a}\right)\cos\frac{2\pi x}{l}+O(\epsilon^2). \tag{18}$$

The second velocity component $v(x,y)$ follows from the continuity equation (9) together with the second boundary condition (12).

4. Constitutive equation and perturbation method for viscoelastic fluids

The aim of the following considerations is to determine the pressure–rate-of-flow characteristic (7) for peristaltic pumping ($\Phi > 0$) with counterpressure ($K > 0$) of an arbitrary simple fluid in the sense of Noll. Under the restrictive assumption of small amplitude of the corrugation ($|\epsilon| \ll 1$) this succeeds with analytical methods. Within the frame of the desired ϵ^2 approximation, the right-hand side of (7) can be replaced by linear expressions in ϵ^2 and K. Since a constant term cannot occur for obvious reasons, the resulting expression consists of two terms only:

$$\Phi = \phi_0(\omega\lambda)\,\epsilon^2+\phi_1 K. \tag{19}$$

The coefficient ϕ_0 should be a function of the memory parameter $\omega\lambda$. In the case of ϕ_1 a dependence on $\omega\lambda$ can be excluded, because in an uncorrugated pipe ($\epsilon = 0$) a constant pressure gradient leads to a steady viscometric flow, for which it is known that memory effects play no role. Thus, the coefficient ϕ_1 is the same as for a Newtonian fluid (without memory). The result (17) gives us the value $\phi_1 = -\frac{1}{12}$. From (19) we see that for an arbitrary viscoelastic fluid in a channel with small amplitude of the corrugation the pressure–discharge characteristics are parallel straight lines in the (K, Φ)-diagram (cf. figure 4). The aim of the following is to connect the coefficient $\phi_0(\omega\lambda)$ with measurable material properties. Since $\Phi/\Phi_0 = 1 - K/K_0$ for sufficiently small values of ϵ, and, since $\phi_1 = -\frac{1}{12}$, we get $K_0 = 12\Phi_0$, a relation that is completely independent of the material properties of the fluid. Because the quantity $\Phi_0 = \phi_0(\omega\lambda)\,\epsilon^2$ only is unknown, it is sufficient for further considerations to study the special situation without global pressure rise.

In order to find the adequate constitutive equation for simple fluids with memory, we have to take into account that in the case $0 < |\epsilon| \ll 1$ we have a motion with small deformation amplitude. Within a theory quadratic in ϵ the most general constitutive equation for incompressible simple fluids is the so-called 'constitutive equation of second-order viscoelasticity'. It reads (Truesdell & Noll 1965)

$$\mathbf{T} = \int_0^\infty \frac{dG(s)}{ds}\,\mathbf{E}(s)\,ds + \int_0^\infty\int_0^\infty \{\alpha(s_1, s_2)\,\mathbf{E}(s_1)\,\mathbf{E}(s_2) + \beta(s_1, s_2)\,[\mathrm{tr}\;\mathbf{E}(s_1)]\,\mathbf{E}(s_2)\}\,ds_1\,ds_2. \tag{20}$$

Here $\mathbf{T}$ denotes the extra stress tensor and $\mathbf{E}(s) \equiv \mathbf{C}_t(s) - \mathbf{I}$ the history of the relative right Cauchy–Green tensor $\mathbf{C}_t$ reduced by the unit tensor $\mathbf{I}$. $\mathbf{C}_t$ is connected according to $\mathbf{C}_t(s) := \mathbf{F}_t^T(s)\,\mathbf{F}_t(s)$ with the relative deformation gradient $\mathbf{F}_t$, which transforms a material line element $d\mathbf{r}$ at the actual time t into its position $d\mathbf{r}^*$ at the former time $t-s$, i.e. $d\mathbf{r}^* = \mathbf{F}_t(s)\,d\mathbf{r}$. $G(s)$ is known as the linear viscoelastic influence function. The material properties $\alpha(s_1, s_2)$ and $\beta(s_1, s_2)$ will play no role in our case and therefore need not be considered in detail. The double integral on $\beta(s_1, s_2)$ can be omitted completely, because according to Pipkin (1964) (see also Huilgol 1975) the linear invariant tr $\mathbf{E}$ disappears like ϵ^2 for every motion with $|\mathbf{E}| = O(\epsilon)$ in an incompressible fluid. Thus the integral represents an additional term of third order, which can be neglected within the scope of an ϵ^2 approximation. Therefore we can formally put $\beta = 0$. The reason why the integral of $\alpha(s_1, s_2)$ also does not affect the motion can be explained only later on.

In order to calculate the flow field in an ϵ^2 approximation it is convenient to use series expansions in powers of ϵ, for instance

$$\frac{u}{c} = 1 + \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3), \quad \frac{v}{c} = \epsilon v_1 + \epsilon^2 v_2 + O(\epsilon^3), \tag{21}$$

$$\tilde{p} = \epsilon\tilde{p}_1 + \epsilon^2\tilde{p}_2 + O(\epsilon^3). \tag{22}$$

Note that the coefficient functions u_i and v_i ($i = 1, 2$) are dimensionless because of the normalization with the wave velocity c. Similarly, the coefficients of the expansions of the relative right Cauchy–Green tensor and of the extra stress tensor are called E_i and T_i respectively.

5. The first-order flow field

The terms that are linear in ϵ are found within the theory of linear viscoelasticity, where the right-hand side of the constitutive equation (20) reduces to the first integral

$$\mathbf{T} = \int_0^\infty \frac{dG(s)}{ds} \mathbf{E}(s)\, ds. \tag{23}$$

As has been shown in §3 the velocity field is independent of the material properties, and therefore corresponds to the Newtonian first-order flow field. The coefficient function $u_1(x, y)$ can thus be read from (18); $v_1(x, y)$ then follows from the equation of continuity (9):

$$u_1 = f'(y) \cos\frac{2\pi x}{l}, \quad v_1 = \frac{2\pi}{l} f(y) \sin\frac{2\pi x}{l}. \tag{24}$$

According to (18), $f'(y) = 6(y^2/a^2 - y/a)$; the prime denotes differentiation with respect to y. The original function that is compatible with the boundary condition $f(0) = 0$ (cf. (12)) reads

$$f(y) = a\left(2\frac{y^3}{a^3} - 3\frac{y^2}{a^2}\right). \tag{25}$$

Since the motion is known in the ϵ^1 approximation, the deformation history $\mathbf{E}_1(s)$ can be calculated. We find

$$\mathbf{E}_1(s) = -\sin\omega s\, \mathbf{A} + (1 - \cos\omega s)\, \mathbf{B}, \tag{26}$$

where $\mathbf{A}$ and $\mathbf{B}$ are respectively the imaginary and real parts of the complex matrix

$$\mathbf{B} + i\mathbf{A} \equiv \begin{bmatrix} -2f' & i\frac{l}{2\pi} f'' \\ i\frac{l}{2\pi} f'' & 2f' \end{bmatrix} \exp\left(i\frac{2\pi x}{l}\right). \tag{27}$$

By substituting the expression (26) into the constitutive equation (23), we obtain the integrals

$$\eta'(\omega) = \int_0^\infty G(s) \cos\omega s\, ds, \quad \eta''(\omega) = \int_0^\infty G(s) \sin\omega s\, ds, \tag{28}$$

which are respectively the real and imaginary parts of the so-called complex viscosity $\eta^*(\omega) \equiv \eta'(\omega) - i\eta''(\omega)$ of the fluid. In this way we find for the first-order extra stresses

$$\mathbf{T}_1 = \omega\eta'(\omega)\, \mathbf{A} - \omega\eta''(\omega)\, \mathbf{B}. \tag{29}$$

If we introduce the expressions for the shear stress and the normal-stress difference into the equation of motion (11) and make use of the slenderness condition again, we get for the first-order pressure field

$$\frac{d\tilde{p}_1}{dx} = c\left[\eta'(\omega) \cos\frac{2\pi x}{l} + \eta''(\omega) \sin\frac{2\pi x}{l}\right]\frac{d^3 f}{dy^3}. \tag{30}$$

According to (25) the factor d^3f/dy^3 is constant and equal to $12/a^2$. Thus not only the velocity field but also the stress field is completely determined to first-order approximation.

6. The induced second-order stresses

One can show that the product $\mathbf{E}_1(s_1)\mathbf{E}_1(s_2)$ is proportional to the unit tensor. This is why we can add, within a second-order theory, the contributions resulting from the double integral on $\mathbf{E}(s_1)\mathbf{E}(s_2)$ in (20) to the pressure field. Therefore it is also possible to take the material function $\alpha(s_1, s_2)$ as zero. Thus we find the unexpected result that even within a second-order approximation the constitutive equation (23) of finite linear viscoelasticity is appropriate.

The next step consists of determining the second-order stresses, which are induced by the first-order motion. These stress contributions will produce an additional motion (the terms u_2, v_2 in (21)), which will influence the stress field again. Thus, it is convenient to split up the total second-order stresses, $\mathbf{T}_2$ and $\tilde{p}_2$, into a sum of two terms in each case:

$$\mathbf{T}_2 = \mathbf{T}^{(1,1)} + \mathbf{T}^{(2)}, \quad \tilde{p}_2 = \tilde{p}^{(1,1)} + \tilde{p}^{(2)}. \tag{31}$$

The indices (1, 1) and (2) indicate that the corresponding quantities are connected with the square of the first-order flow field, or linearly with the second-order flow field. In order to get the fields $\mathbf{T}^{(1,1)}$ and $\tilde{p}^{(1,1)}$, the contribution $\mathbf{E}^{(1,1)}$ to the deformation history has to be determined. The required calculations are rather extensive, and are omitted here. Introducing the result into (23) we meet once more the integrals in the form (28). However, apart from the quantities $\eta'(\omega), \eta''(\omega)$, the viscosity values $\eta'(2\omega)$ and $\eta''(2\omega)$ corresponding to the double frequency 2ω also appear, because the second-order deformation includes contributions that oscillate with 2ω. Moreover, the integral

$$-\int_0^\infty \frac{dG(s)}{ds} s\, ds = \int_0^\infty G(s)\, ds = \eta_0 \tag{32}$$

occurs, which is known to agree with the zero-shear-rate viscosity of the fluid (cf. Böhme 1981). So we find the stress components

$$\tau_{xy}^{(1,1)} = -\frac{c}{2}[\eta_0 - \eta'(\omega)]\,(ff')'' + \frac{c}{2}\left\{[\eta'(\omega) - \eta'(2\omega)] \cos\frac{4\pi x}{l} + [\eta''(\omega) - \eta''(2\omega)] \sin\frac{4\pi x}{l}\right\}(ff'' - 2f'^2)', \tag{33}$$

$$\tau_{xx}^{(1,1)} - \tau_{yy}^{(1,1)} = \frac{cl}{2\pi}\left\{\eta''(\omega) - [\eta''(\omega) - \eta''(2\omega)] \cos\frac{4\pi x}{l} + [\eta'(\omega) - \eta'(2\omega)] \sin\frac{4\pi x}{l}\right\} f''^2. \tag{34}$$

In the equation of motion (11) the sum of the stress derivatives

$$\frac{\partial \tau_{xy}^{(1,1)}}{\partial y} + \frac{\partial}{\partial x}[\tau_{xx}^{(1,1)} - \tau_{yy}^{(1,1)}] = -\frac{c}{2}[\eta_0 - \eta'(\omega)]\,(ff')''' + \frac{c}{2}\left\{[\eta'(\omega) - \eta'(2\omega)] \cos\frac{4\pi x}{l} + [\eta''(\omega) - \eta''(2\omega)] \sin\frac{4\pi x}{l}\right\}(f''^2 - 2f'f''') \tag{35}$$

occurs. Because of the special form of the function $f(y)$ (cf. (25)) the factor $f''^2 - 2f'f'''$ is constant and equal to $36/a^2$. Thus the terms multiplied by $\cos(4\pi x/l)$ and $\sin(4\pi x/l)$ are independent of y, and therefore can be counterbalanced by an appropriate pressure field $\tilde{p}^{(1,1)}$. The remaining term $-\frac{1}{2}c[\eta_0 - \eta'(\omega)]\,(ff')'''$ can be considered to represent a non-conservative body force in the x-direction. It induces a motion of second order (the terms u_2, v_2 in (21)), which produces additional second-order stresses (the terms $\mathbf{T}^{(2)}$ and $\tilde{p}^{(2)}$ in (31)). The aim of §7 is to determine this second-order flow field and to describe how to get the corresponding stresses.

7. The second-order flow field

First of all we turn to the boundary conditions, which constrain the second-order velocity field. From (12) it follows that

$$u_2(x,0) = 0, \quad v_2(x,0) = 0. \tag{36}$$

Making use of the expression (21), as well as of (6) and (24), a systematic series expansion of the conditions (13) with respect to ϵ leads to

$$u_2(x,a) = -3\left(1+\cos\frac{4\pi x}{l}\right), \quad v_2(x,a) = 0. \tag{37}$$

To determine the fields $u_2(x,y)$ and $v_2(x,y)$ a theory linear in u_2, v_2 using the constitutive equation (23) suffices. Since the driving body force depends on y only, and because of the special form of the boundary conditions, we suppose that the x-component of the velocity field under discussion has the form

$$u_2 = w(y)+\hat{f}'(y)\cos\frac{4\pi x}{l}. \tag{38}$$

It follows from continuity that the y-component then reads

$$v_2 = \frac{4\pi}{l}\hat{f}(y)\sin\frac{4\pi x}{l}. \tag{39}$$

Equations (36) and (37) lead to the following boundary conditions for the two unknown functions $w(y)$ and $\hat{f}(y)$:

$$w(0) = 0, \quad w(a) = -3, \tag{40}$$

$$\hat{f}(0) = \hat{f}'(0) = \hat{f}(a) = 0, \quad \hat{f}'(a) = -3. \tag{41}$$

According to (38) and (39) the second-order flow field consists of a steady shear flow and a harmonically oscillating motion. The latter is calculated within the linear viscoelastic theory analogously to the first-order flow field (notice the affinity with (24)); we have only to substitute $\frac{1}{2}l$ for l, that is ω by 2ω as well as $f(y)$ by $\hat{f}(y)$. In particular, we can use the result (30) for the pressure field again:

$$\frac{d\tilde{p}^{(2)}}{dx} = c\left\{\eta'(2\omega)\cos\frac{4\pi x}{l}+\eta''(2\omega)\sin\frac{4\pi x}{l}\right\}\frac{d^3\hat{f}}{dy^3}. \tag{42}$$

Since the pressure depends on x only, $\hat{f}'''$ has to be constant, i.e. $\hat{f}^{\text{iv}} = 0$. The solution of this differential equation, which obeys the boundary conditions (41), reads

$$\hat{f}(y) = 3a\left(\frac{y^2}{a^2}-\frac{y^3}{a^3}\right). \tag{43}$$

Concerning the velocity contribution $w(y)$, it is clear that a steady plane shear flow of a linear viscoelastic fluid with the velocity field $cw(y)$ [m/s] is connected with the shear stress $c\eta_0 dw/dy$ [N/m^2]. The force balance on a fluid element connects its spatial derivative with the driving body force:

$$c\eta_0\frac{d^2w}{dy^2}-\tfrac{1}{2}c\,[\eta_0-\eta'(\omega)]\,(ff')''' = 0. \tag{44}$$

With the boundary conditions (40) and subject to special properties of the function $f(y)$, (44) yields

$$w(y) = -3\frac{\eta'(\omega)}{\eta_0}\frac{y}{a}+\frac{1}{2}\left[1-\frac{\eta'(\omega)}{\eta_0}\right](ff')'. \tag{45}$$

This determines the second-order velocity field completely. It is influenced by a single material constant, namely $\eta'(\omega)/\eta_0$, and this concerns only the viscometric part $w(y)$. The harmonically oscillating parts of first and second order are completely independent of the material properties. Note therefore that in the case of a viscoelastic fluid even in an ϵ^2 approximation the velocity component v is the same as in the case of a Newtonian fluid. As regards the extra stresses and the pressure, this is naturally not true. On the contrary, as (33) and (34) already suggest, the material constants η_0, $\eta'(\omega)$, $\eta''(\omega)$, $\eta'(2\omega)$ and $\eta''(2\omega)$ enter the result. Since the complete second-order stress field will not be needed, the explicit formulas for the contributions $\tau_{xy}^{(2)}$ and $\tau_{xx}^{(2)}-\tau_{yy}^{(2)}$ can be omitted here.

8. Rate of flow and efficiency

In order to come back to the aspects described in § 1 we calculate the flow rate by integration of the velocity field. Expanding consistently in powers of ϵ we obtain within an ϵ^2 approximation

$$q_0 = ca+\epsilon^2 c\int_0^a w(y)\,dy = ca-\tfrac{3}{2}\epsilon^2 ca\frac{\eta'(\omega)}{\eta_0}. \tag{46}$$

The subscript 0 used with the symbol q recalls that this is the flow rate without counterpressure (for $K = 0$). From (46) it is evident that all harmonically oscillating parts of the motion do not influence the flow rate. The relation (8) provides the time-averaged discharge $\bar{Q}_0$ in the laboratory frame, which is more relevant for applications. Concerning the corresponding dimensionless flow-rate parameter (cf. the definition (1)) we find the remarkably simple result

$$\Phi_0 = \tfrac{3}{2}\epsilon^2\frac{\eta'(\omega)}{\eta_0}. \tag{47}$$

With this the coefficient function $\phi_0(\omega\lambda)$, which was unknown in (19), is now determined. Apart from the numerical factor $\frac{3}{2}$ it agrees with the ratio of the dynamic viscosity $\eta'(\omega)$ and the zero-shear-rate viscosity η_0. In fluid-polymer systems (solutions and melts) $\eta'(\omega)/\eta_0$ decreases with frequency. The model of a Maxwell body with relaxation time λ which obeys the relation

$$\frac{\eta'(\omega)}{\eta_0} = \frac{1}{1+(\omega\lambda)^2} \tag{48}$$

fits real material data, at least qualitatively. According to (47) and (48) the flow-rate parameter Φ_0 (without counterpressure) and the maximum sustainable value of the pressure parameter K_0 (remember the relation $K_0 = 12\Phi_0$ between Φ_0 and K_0) decrease monotonically with the memory parameter $\omega\lambda$; cf. figure 2. For the physical quantities $\bar{Q}_0$ and $\Delta p_{l,0}$ the following consequences arise. While for a Newtonian fluid $\bar{Q}_0$ and $\Delta p_{l,0}$ are proportional to the wave velocity c, in the case of a viscoelastic fluid both quantities increase only until the memory parameter $\omega\lambda$ reaches a certain value. For a Maxwell fluid this critical value is 1. In this situation the memory already extends over several wave periods. If we increased the wave velocity further, $\bar{Q}$ and

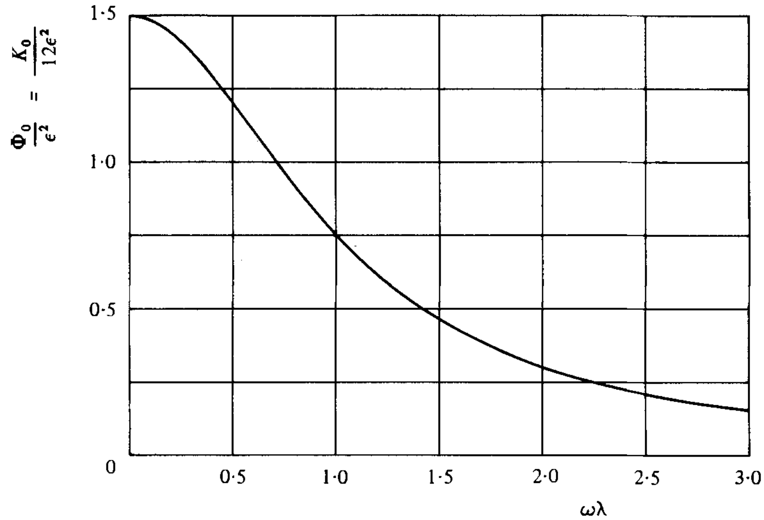

FIGURE 2. Maximum rate of discharge and maximum counterpressure in dependence of the memory parameter; Maxwell model.

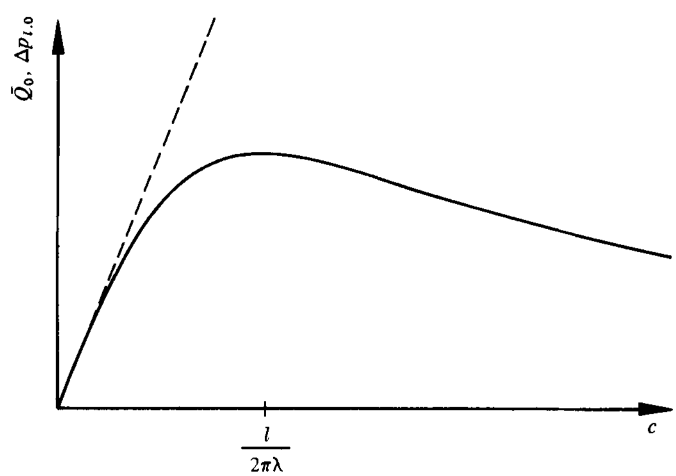

FIGURE 3. Connection between $\bar{Q}_0$, $\Delta p_{l,0}$ and c (sketch only): ---, Newtonian fluid; —, viscoelastic fluid.

$\Delta p_{l,0}$ would decrease again (cf. figure 3). With regard to the greatest possible flow rate, $c = l/2\pi\lambda$ would be the optimum wave velocity for a Maxwell fluid. The elasticity of the fluid and therefore its memory of former states of deformation cause considerable deviations as compared with the inelastic case.

In §4 we showed that the pressure–discharge characteristics for peristaltic pumping are described by the linear relation $\Phi = \Phi_0 - \frac{1}{12}K$ if the amplitude of the corrugation is sufficiently small. By use of (47) we now get the explicit result

$$\Phi = \tfrac{3}{2}\epsilon^2 \frac{\eta'(\omega)}{\eta_0} - \frac{K}{12}. \tag{49}$$

Figure 4 illustrates this formula again for a Maxwell fluid.

To estimate the virtues of peristalsis we calculate the energetic efficiency $\bar{\eta}$ as the ratio of the useful power and the applied power. Because of the periodicity of the

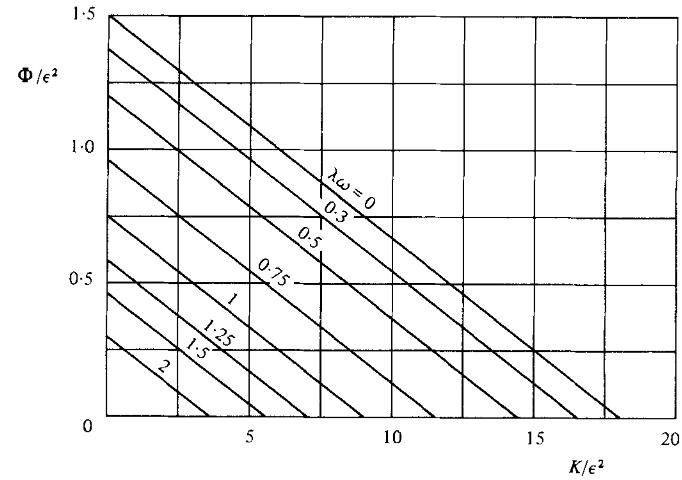

FIGURE 4. Normalized pressure–flow-rate characteristics; Maxwell model.

flow it suffices to consider an element of the flow channel of one wavelength. If we normalize the useful power, that is the product of the flow rate $\bar{Q}$ and the pressure rise Δp_l, according to (1) and (2), we get ΦK, which represents a quantity of order ϵ^4, because both factors are of order ϵ^2. The applied power as sum of the useful power and of the dissipation power contains contributions of order ϵ^2, which originate from the first-order motion. The efficiency is therefore of order ϵ^2, and can be calculated, within the frame of the approximation, as the ratio of the useful power $\bar{Q}\Delta p_l$ and the dissipation power connected with the first-order flow field:

$$\bar{\eta} = \frac{\bar{Q}\Delta p_l}{\epsilon^2 \int_0^a \int_0^l \operatorname{tr}\,[\mathbf{T}_1 \mathbf{D}_1]\, dx\, dy}. \tag{50}$$

($\mathbf{D}$ denotes the rate-of-deformation tensor, the symmetric part of the velocity gradient.) Using the explicit results from § 5 the denominator in (50) can be reduced to the expression $6\epsilon^2 c^2 \eta'(\omega)\, l/a$. So we find for the efficiency $\bar{\eta} = \Phi K/[6\epsilon^2 \eta'(\omega)/\eta_0]$. Eliminating Φ by the relation (49), the final formula

$$\frac{\bar{\eta}}{\epsilon^2} = \frac{1}{4}\frac{K}{\epsilon^2} - \frac{1}{72}\frac{\eta_0}{\eta'(\omega)}\left(\frac{K}{\epsilon^2}\right)^2 \tag{51}$$

results. It shows that in the case of an arbitrary viscoelastic fluid the efficiency is a quadratic function of the pressure parameter. It is clear that $\bar{\eta}$ vanishes if one of the two factors in the numerator of (50) disappears, i.e. for $K = 0$ and for $K = K_0$ (if $\Phi = 0$). The greatest possible efficiency is attained for $K = \frac{1}{2}K_0$ (that means at the same time $\Phi = \frac{1}{2}\Phi_0$) and has the value $\bar{\eta}_{max} = \frac{9}{8}\epsilon^2\eta'(\omega)/\eta_0 = \frac{3}{4}\Phi_0$. For real viscoelastic fluids, whose dynamic viscosity decreases monotonically with the frequency (cf. figure 2), the maximum attainable efficiency decreases as the frequency parameter $\omega\lambda$ increases. With these considerations, figure 5, which illustrates the analytical result (51), is easy to understand. Thus, from an energetic point of view those wave velocities c are the best that are small compared with $l/2\pi\lambda$, a value that had special significance in figure 3. The viscoelastic fluid then changes its state so slowly that the memory, and with it the elasticity, have no influence at all ($\omega\lambda \ll 1$). The Newtonian limit ($\omega\lambda = 0$) therefore represents an upper bound for the efficiency. On the other hand,

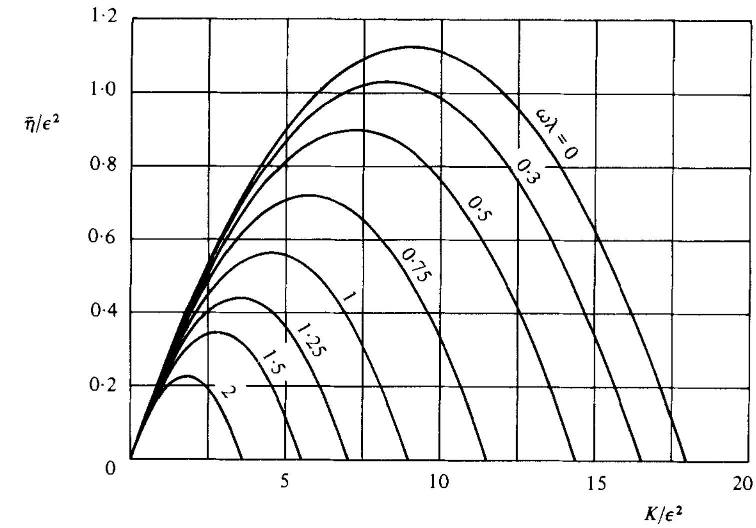

FIGURE 5. Efficiency as function of the pressure parameter and the memory parameter; Maxwell model.

for a fluid whose dynamic viscosity increases with the frequency, the Newtonian limit would be a lower bound, i.e. the efficiency could be improved by increasing the wave velocity.

9. Conclusions

The plane peristaltic flow of an incompressible viscoelastic fluid is described by five independent dimensionless quantities: the Reynolds number R, the dimensionless wavenumber a/l, the amplitude ratio ϵ, the memory parameter $\omega\lambda$ and the pressure parameter K. On the assumption that the reduced Reynolds number is small enough ($Ra/l \rightarrow 0$), inertia forces are neglected. The assumption $a/l \ll 1$ has the consequence that the pressure is constant over the channel height. Restricting to sufficiently small values of ϵ the flow field can be approximated by the first terms of a series in ascending powers of ϵ. Within the frame of a second-order approximation the integral constitutive equation of finite linear viscoelasticity still applies. The theoretical analysis gives explicit expressions for the velocity fields, the pressure and the extra stresses. Particular interest is directed to the rate of flow produced by the peristaltic motion, and at the pumping efficiency, for which simple analytical results are deduced and discussed ((49) and (51)). The results are influenced by specific values of the complex viscosity of the fluid, which depend on the magnitude of the memory parameter $\omega\lambda$.

In regard to the applications in biomechanics and physiology the two assumptions $Ra/l \rightarrow 0$ and $a/l \ll 1$, which cause remarkable simplifications, may be considered to be realistic. But it seems desirable to weaken the third assumption $|\epsilon| \ll 1$ in order to describe peristaltic waves of arbitrary amplitude ($|\epsilon| < 1$). General statements for simple fluids are then possible only under restrictive conditions, in particular for sufficiently slow flow, when the memory of the fluid particles reaches back only a short part of a wave period, $\omega\lambda \ll 1$. Otherwise a specific fluid model has to be chosen in order to study the peristaltic transport in the case of arbitrary wave amplitude. This will be discussed by the second author elsewhere.

REFERENCES

BECKER, E. 1980 Simple non-Newtonian fluid flows. *Adv. Appl. Mech.* **20**, 177–226.

BÖHME, G. 1981 *Strömungsmechanik nicht-newtonscher Fluide*. Teubner.

HUILGOL, R. R. 1975 *Continuum Mechanics of Viscoelastic Liquids*. Wiley.

PIPKIN, A. C. 1964 Small finite deformations of viscoelastic solids. *Rev. Mod. Phys.* **36**, 1034–1041.

RAJU, K. K. & DEVANATHAN, R. 1972 Peristaltic motion of a non-Newtonian fluid. *Rheologica Acta* **11**, 170–178.

RAJU, K. K. & DEVANATHAN, R. 1974 Peristaltic motion of a non-Newtonian fluid; Part II, Visco-elastic fluid. *Rheologica Acta* **13**, 944–948.

RATH, H. J. 1980 *Peristaltische Strömungen*. Springer.

SHAPIRO, A. H., JAFFRIN, M. Y. & WEINBERG, S. L. 1969 Peristaltic pumping with long wavelengths at low Reynolds number. *J. Fluid Mech.* **37**, 799–825.

SHUKLA, J. B., PARIHAR, R. S., RAO, B. R. P. & GUPTA, S. P. 1980 Effects of peripheral-layer viscosity on peristaltic transport of a bio-fluid. *J. Fluid Mech.* **97**, 225–237.

TRUESDELL, C. & NOLL, W. 1965 The non-linear field theories of mechanics. *Encyclopedia of Physics* (ed. S. Flügge), vol. III/3. Springer.